SOLUTION OF PROBLEM 89. (SEE PAGE 195, VOL. II.)

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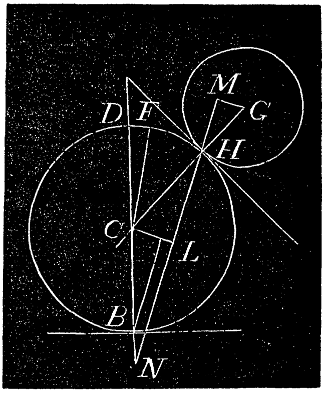
LET m, m', be the masses of the spheres R and r; x, y, the horizontal and vertical coordinates of r's centre with reference to R's centre when in its initial position, $\theta = DCG =$ angle between line of centres and vertical at time t, θ_1, θ_2 = angular rotations of R and r about centres, a = initial value of θ, F = magnitude and direction of the resultant of the reaction between the spheres at the point of contact H, p = the magnitude and direction of the normal pressure at H.



From the geometrical relations, $x = R\theta_1 + (R+r)\sin\theta \ldots (1)$, $y = (R+r)\cos\theta, \ldots (2)$ $R(\theta - a - \theta_1) =$ extent of arcs bro't into contact, $R(\theta - a - \theta_1) \div r + \theta - a = [(R+r)(\theta - a) - R\theta_1] \div r = \theta_2 \ldots (3)$, $dx^2 + dy^2 = (R+r)d\theta^2 + R^2 d\theta_1^2 + 2R(R+r)\cos\theta d\theta d\theta_1$, (4)

There being no loss of vis viva in perfect rolling, $2m'g(R+r)(\cos a - \cos\theta)$

$$= m'\frac{dx^2+dy^2}{dt^2} + \tfrac{2}{5}m'r^2\frac{d\theta_2^2}{dt^2} + \tfrac{7}{5}mR^2\frac{d\theta_1^2}{dt^2} = \tfrac{7}{5}m'(R+r)^2\frac{d\theta^2}{dt^2} + \tfrac{7}{5}(m+m')R^2\frac{d\theta_1^2}{dt^2}$$

$+ 2R(R+r)(\cos\theta - \frac{2}{5})m'\frac{d\theta d\theta_1}{dt^2} \ldots (5)$; $F\sin\frac{p}{F} =$ tangential component of F at H, $F\cos\frac{p}{F} = p = m'g\cos\theta - m'(R+r)\frac{d\theta^2}{dt^2} \ldots (6)$, = normal pressure at H, = normal component of weight of r minus its centrifugal force.

$F\sin\left(\theta - \frac{p}{F}\right) =$ horizontal component of F at H, acting to right on lower sphere, also = horizontal force at B, the lower point of contact. F, acting in direction HN, is the only force communicated by the weight of the upper sphere to the lower, and therefore the lower sphere will tend to move to the right or to the left according as HN passes to the right or left of B.

By the principle that the angular acceleration of a body about a fixed axis = the moment of the impressed forces divided by the moment of inertia with respect to that axis (Bartlett's An. Mech. 8th ed. p. 248), I have the following four eq'ns: The angular acceleration of r about its central axis = that which would be produced by the force F acting upward with the lever arm GM, while the body were retained by a fixed axis through G, hence

$Fr\sin\frac{p}{F} \div \frac{2}{5}m'r^2 = \frac{d^2\theta_2}{dt^2} \ldots (7)$; $FR\left(\sin\frac{p}{F} - \sin(\theta - \frac{p}{F})\right) \div \frac{7}{5}mR^2 = \frac{d^2\theta_1}{dt^2}, \ldots (8)$ for lower sphere about axis through B.

For common rotation of both spheres about axis through centre of lower,

$$FR\sin\left(\theta - \frac{p}{F}\right) \div \left\{ \tfrac{2}{5}mR^2 + m'[\tfrac{2}{5}r^2 + (R+r)^2] \right\} = \frac{d^2\theta_1}{dt^2}, \quad \ldots\ldots (9)$$

and for rotation of upper sphere about axis through H,

$$r\left(m'g\sin\theta - m'(R+r)\frac{d^2\theta_1}{dt^2}\right) \div \tfrac{7}{5}m'r^2 = \frac{d^2\theta_2}{dt^2}. \quad \ldots\ldots (10)$$

Eliminating $F\sin\frac{p}{F}$ and $F\sin\left(\theta - \frac{p}{F}\right)$ from (8) by (7) and (9),

$$\tfrac{2}{5}m'r\frac{d^2\theta_2}{dt^2} = \frac{9mR^2 + m'[2r^2 + 5(R+r)^2]}{5R}\cdot\frac{d^2\theta_1}{dt^2}. \quad \ldots\ldots (11)$$

Integrating twice and observing that θ_1 and θ_2 begin together,

$$\theta_2 = \frac{(9m + 5m')R^2 + 7m'r^2 + 10m'rR}{2m'rR} \times \theta_1. \quad \ldots\ldots (12)$$

Hence by (3) $\theta_1 = \dfrac{2m'R(r+R)(\theta - a)}{(9m + 7m')R^2 + 7m'r^2 + 10m'rR}$. ....(13)

Substituting in (1) θ_1 from (13) and θ from (2)

$x = 2m'R^2(R+r)(\theta - a) \div [9mR^2 + m'(7R^2 + 7r^2 + 10rR)] + (R+r)\sin\theta$
$= 2m'R^2(R+r)\left\{[\cos^{-1} y \div (R+r) - a]\right\} \div [9mR^2 + m'(7R^2 + 7r^2$
$+ 10rR)] + \sqrt{[(R+r)^2 - y^2]}$ ... (14), which is the required equation.

From the figure $NC : R :: \sin CHN : \sin N$; hence by (7) and (9)

$$NC : R :: \tfrac{2}{5}m'r\frac{d^2\theta_2}{dt^2} : \frac{\tfrac{2}{5}mR^2 + m'[\tfrac{2}{5}r^2 + (R+r)^2]}{R}\,\frac{d^2\theta_1}{dt^2}, \quad \text{and}$$

by (11) $NC : R :: 9mR^2 + m'[2r^2 + 5(R+r)^2] : 2mR^2 + m'[2r^2 + 5(R+r)^2]$, in which the first term of second couplet being greater than the second term by $7mR^2$, NC is in all cases greater than R, and therefore F or HN passes to the right of the point of contact B, and both spheres in all cases have their motions on the same side of the origin*.

The value of $d\theta_1$ from (13) in (5) gives $\frac{d\theta^2}{dt^2}$, which in (6), $p = m'g\cos\theta - (R+r)m'\frac{d\theta^2}{dt^2} = 0$, gives an equation of the 2nd degree in $\cos\theta$, from which one of the values of $\cos\theta$ gives the point of separation.

By equality of (7) and (10), $F\sin\frac{p}{F} = \tfrac{2}{7}m'g\sin\theta - \tfrac{2}{7}m'\frac{R+r}{r}\cdot\frac{d^2\theta_1}{dt^2}$, which shows that the tangential component employed in giving rotation to upper sphere about its central axis $= \frac{2}{7}$ of the tangential component of the weight $m'g$, only when the surface on which the rolling takes place has no motion.

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*We dissent from this conclusion, 1st: Because it may easily be shown, from a different course of reasoning, that, in certain positions, the spheres will roll in opposite directions;